

Conditions for entanglement in multipartite systems

Mark Hillery¹, Ho Trung Dung^{1,2}, Hongjun Zheng¹

⁽¹⁾Department of Physics, Hunter College of CUNY
695 Park Avenue
New York, NY 10065

⁽²⁾ Institute of Physics, Academy of Sciences and Technology
1 Mac Dinh Chi Street, District 1
Ho Chi Minh City, Vietnam

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Abstract

We introduce two entanglement conditions that take the form of inequalities involving expectation values of operators. These conditions are sufficient conditions for entanglement, that is if they are satisfied the state is entangled, but if they are not, one can say nothing about the entanglement of the state. These conditions are quite flexible, because the operators in them are not specified, and they are particularly useful in detecting multipartite entanglement. We explore the range of utility of these conditions by considering a number of examples of entangled states, and seeing under what conditions entanglement in them can be detected by the inequalities presented here.

1 Introduction

Besides being of fundamental interest, entanglement among more than two parties can potentially be an important resource in quantum communication and information processing [1, 2]. Quantum teleportation, quantum dense coding, quantum telecloning and quantum key distribution schemes involving two parties are extendible to an arbitrary number of parties sharing multipartite entanglement. Further proposals that exploit the multiparty quantum correlations of multipartite entangled states include quantum secret sharing, where parties may share quantum information retrievable only when all parties cooperate [3], remote concentration of quantum information [4], and measurement-based quantum computing [5].

The structure of entanglement in multipartite systems is much richer than that in the case of bipartite systems. Despite the fact that considerable effort has been spent on characterizing multipartite entanglement, the detection, classification, and quantification of entanglement for arbitrary states of multipartite systems remains a formidable task [1, 2]. In this paper we focus on the problem of detecting entanglement in multipartite systems using inequalities. One possible strategy in this approach is to use pairwise inequalities to check for entanglement in every possible bipartite cut in the system. In this way one may gain detailed information about which subsystems are entangled [6, 7]. However, the amount of work required to perform the task can grow enormously as the

number of subsystems increases. It is desirable to have multipartite inequalities that would allow one to check for overall entanglement in multipartite systems in a straightforward and transparent manner. For systems of n qubits, inequalities of this type exist [8]-[14]. These inequalities typically involve collective spin operators, and are simple to apply.

We shall present two inequalities in this paper that detect the presence of entanglement in multipartite systems. These are an outgrowth of earlier work on entanglement in continuous-variable systems. Within the last few years, several papers have presented inequalities for detecting entanglement in two-mode continuous-variable systems, which are particularly useful for non-Gaussian states [6, 7],[15] - [19]. We note that the papers [6, 7, 19] dealt with multipartite entanglement. The inequalities are sufficient conditions for entanglement, if they are satisfied, the state is entangled, but if they are not, nothing can be concluded. In most cases these inequalities can be derived from the partial transpose condition, though the inequalities in [15] were not originally proved in this way (see [18, 20]). In fact, the inequalities in [15] provide sufficient conditions to detect entanglement in any bipartite system, not just in continuous-variable systems, and they have been applied to explore entanglement in two-mode field states [6], spin systems [21], and atom-field entanglement [20].

Let us now state the entanglement conditions for multipartite systems, which are the subject of this paper. Suppose we have a system consisting of n subsystems, and let A_k be an operator on the Hilbert space of the k^{th} subsystem. A state is entangled if either of the two conditions

$$\left| \left\langle \prod_{k=1}^n A_k \right\rangle \right| > \prod_{k=1}^n \langle (A_k^\dagger A_k)^{n/2} \rangle^{1/n}, \quad (1)$$

$$\left| \left\langle \prod_{k=1}^n A_k \right\rangle \right| > \left\langle \left(\frac{1}{n} \sum_{k=1}^n A_k^\dagger A_k \right)^{n/2} \right\rangle, \quad (2)$$

is satisfied. These inequalities are applicable to systems of continuous-variable type, discrete type, or a mixture between the two. We shall first prove these inequalities, and then proceed to discuss their consequences by making use of several examples.

2 Separability Conditions

Consider a system consisting of n subsystems with Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_n$. If the system is in a pure state, it is fully separable if and only if the state is a product of pure states describing n elementary subsystems. If the state is mixed, it is fully separable if ρ is a statistical mixture of product states

$$\rho = \sum_j p_j \rho_j = \sum_j p_j \rho_j^{(1)} \otimes \rho_j^{(2)} \cdots \otimes \rho_j^{(n)}. \quad (3)$$

Let A_k be an operator on \mathcal{H}_k , then we have for a fully separable state that

$$\begin{aligned}
\left| \left\langle \prod_{k=1}^n A_k \right\rangle \right| &= \left| \sum_j p_j \prod_{k=1}^n \langle A_k \rangle_j \right| \\
&\leq \sum_j p_j \left| \prod_{k=1}^n \langle A_k \rangle_j \right| \\
&\leq \sum_j p_j \prod_{k=1}^n \langle |A_k|^2 \rangle_j^{1/2}.
\end{aligned} \tag{4}$$

where $\langle A_k \rangle_j = \text{Tr}(A_k \rho_j)$, and $|A_k|$ denotes $\sqrt{A_k^\dagger A_k}$. In the first line we used the full separability of the state and in going from the second line to the third, we used the fact that any operator has a non-negative variance

$$|\langle A_k \rangle_j| \leq \langle |A_k|^2 \rangle_j^{1/2}.$$

We prove now a lemma.

Lemma: For any positive operator B we have that $\langle B \rangle^p \leq \langle B^p \rangle$, $p > 1$.

Proof: First we write $\langle B \rangle$ in the form

$$\langle B \rangle = \sum_{l=1}^m \lambda_l \langle P_l \rangle, \tag{5}$$

where P_l is the projector corresponding to λ_l and $\langle P_l \rangle = \text{Tr}(\rho P_l)$. We shall make use of the Hölder inequality [22], which is

$$\sum_{l=1}^m |x_l y_l| \leq \left(\sum_{l=1}^m |x_l|^p \right)^{1/p} \left(\sum_{l=1}^m |y_l|^q \right)^{1/q}, \tag{6}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad q > 1, \tag{7}$$

and the equality holds iff $|x_1|^{p-1}/|y_1| = |x_2|^{p-1}/|y_2| = \dots = |x_m|^{p-1}/|y_m|$. For $p = q = 2$ it reduces to the Cauchy-Schwarz inequality. If we set

$$x_l = \lambda_l \langle P_l \rangle^{1/p}, \quad y_l = \langle P_l \rangle^{1/q}, \tag{8}$$

where p and q satisfy Eq. (7), in the Hölder inequality, it follows that

$$\sum_{l=1}^m |\lambda_l \langle P_l \rangle| = \sum_{l=1}^m \lambda_l \langle P_l \rangle^{1/p} \langle P_l \rangle^{1/q} \leq \left(\sum_{l=1}^m \lambda_l^p \langle P_l \rangle \right)^{1/p} \left(\sum_{l=1}^m \langle P_l \rangle \right)^{1/q} = \left(\sum_{l=1}^m \lambda_l^p \langle P_l \rangle \right)^{1/p}, \tag{9}$$

hence $\langle B \rangle \leq \langle B^p \rangle^{1/p}$. ■

2.1 Derivation of condition (1)

We shall employ the generalized Hölder inequality [22], which is

$$\left(\sum_j p_j a_j^r b_j^r \dots l_j^r\right)^{\frac{1}{r}} \leq \left(\sum_j p_j a_j^{r/\alpha}\right)^{\alpha/r} \left(\sum_j p_j b_j^{r/\beta}\right)^{\beta/r} \dots \left(\sum_j p_j l_j^{r/\gamma}\right)^{\gamma/r}, \quad (10)$$

where

$$\sum_j p_j = 1, \quad \alpha + \beta + \dots + \gamma = 1. \quad (11)$$

Setting $r = 1$, $\alpha = \beta = \dots = \gamma = \frac{1}{n}$, and $a_j = \langle |A_1|^2 \rangle_j^{1/2}$, $b_j = \langle |A_2|^2 \rangle_j^{1/2}, \dots, l_j = \langle |A_n|^2 \rangle_j^{1/2}$, the inequality (10) readily yields

$$\begin{aligned} \sum_j p_j \prod_{k=1}^n \langle |A_k|^2 \rangle_j^{1/2} &\leq \prod_{k=1}^n \left(\sum_j p_j \langle |A_k|^2 \rangle_j^{n/2} \right)^{1/n} \\ &\leq \prod_{k=1}^n \left(\sum_j p_j \langle |A_k|^n \rangle_j \right)^{1/n} \\ &= \prod_{k=1}^n \langle |A_k|^n \rangle^{1/n}, \end{aligned} \quad (12)$$

where in the second step we made use of the lemma. This and Eq. (4) lead to

$$\left| \left\langle \prod_{k=1}^n A_k \right\rangle \right| \leq \prod_{k=1}^n \langle (A_k^\dagger A_k)^{n/2} \rangle^{1/n}. \quad (13)$$

Since all fully separable states must satisfy the inequality (13), a state that violates it is an entangled state and we obtain the multipartite entanglement condition (1).

2.2 Derivation of condition (2)

To derive Eq. (2) we make use of the fact that the geometric mean is smaller than or equal to the arithmetic mean

$$\prod_{k=1}^n a_k^{1/n} \leq \frac{1}{n} \sum_{k=1}^n a_k, \quad a_k \geq 0, \quad (14)$$

with equality holding iff $a_1 = a_2 = \dots = a_n$. With $a_k = \langle |A_k|^2 \rangle_j^{1/2}$, the inequality (14) yields

$$\begin{aligned}
\prod_{k=1}^n \langle |A_k|^2 \rangle_j^{1/2} &\leq \frac{1}{n^n} \left(\sum_{k=1}^n \langle |A_k|^2 \rangle_j^{1/2} \right)^n \\
&\leq \frac{1}{n^n} n^{n/2} \left(\sum_{k=1}^n \langle |A_k|^2 \rangle_j \right)^{n/2} \\
&= \frac{1}{n^{n/2}} \left\langle \sum_{k=1}^n |A_k|^2 \right\rangle_j^{n/2} \\
&\leq \frac{1}{n^{n/2}} \left\langle \left(\sum_{k=1}^n |A_k|^2 \right)^{n/2} \right\rangle_j, \tag{15}
\end{aligned}$$

where in going from the first line to the second we applied the Cauchy-Schwarz inequality and in going from the third line to the fourth, we used the result of the lemma with $B = \sum_{k=1}^n |A_k|^2$ and $p = n/2$. The inequality (15) leads to

$$\sum_j p_j \prod_{k=1}^n \langle |A_k|^2 \rangle_j^{1/2} \leq \frac{1}{n^{n/2}} \sum_j p_j \left\langle \left(\sum_{k=1}^n |A_k|^2 \right)^{n/2} \right\rangle_j = \frac{1}{n^{n/2}} \left\langle \left(\sum_{k=1}^n |A_k|^2 \right)^{n/2} \right\rangle. \tag{16}$$

Substituting this in Eq. (4), we arrive at the condition

$$\left| \left\langle \prod_{k=1}^n A_k \right\rangle \right| \leq \frac{1}{n^{n/2}} \left\langle \left(\sum_{k=1}^n A_k^\dagger A_k \right)^{n/2} \right\rangle, \tag{17}$$

which must be obeyed by a fully separable state. Its violation yields the multipartite entanglement condition (2).

An inspection of the conditions (1) and (2) reveals that for states such that $A_k^\dagger A_k |\psi\rangle = A_{k'}^\dagger A_{k'} |\psi\rangle \forall k, k'$, these conditions are the same.

In the case of bipartite systems $n = 2$, the inequality (1) reduces to

$$|\langle AB \rangle|^2 > \langle A^\dagger A \rangle \langle B^\dagger B \rangle, \tag{18}$$

while the second inequality, Eq. (2), becomes $|\langle AB \rangle| > \frac{1}{2}(\langle A^\dagger A \rangle + \langle B^\dagger B \rangle)$ or

$$|\langle AB \rangle|^2 > \langle A^\dagger A \rangle \langle B^\dagger B \rangle + \frac{1}{4}(\langle A^\dagger A \rangle - \langle B^\dagger B \rangle)^2. \tag{19}$$

The bipartite entanglement condition (18) is exactly one of those previously derived in Ref. [15], while the condition (19) is generally weaker than the condition (18) because the second term in the right-hand side is nonnegative. However, without specifying the state of the system, there seems to be no easy way to compare the two conditions for $n > 2$. In fact, as we shall see in the next section, there are situations where the second condition, Eq. (2), can detect entanglement, while the first condition, Eq. (1), cannot.

3 Examples

The two entanglement conditions presented in this paper can be applied to both discrete and continuous systems. We shall present examples of both. These examples illustrate some of the kinds of states for which these conditions can detect entanglement, and also the differences between the two conditions.

3.1 GHZ-type states

3.1.1 Generalized GHZ state

We begin by considering a system consisting of n qubits in the state

$$|\psi\rangle = \cos\theta|0\rangle^{\otimes n} + \sin\theta|1\rangle^{\otimes n}. \quad (20)$$

If we choose A_k to be

$$A_k = |0\rangle_k\langle 1|, \quad A_k^\dagger A_k = |1\rangle_k\langle 1|, \quad (21)$$

then $A_k^\dagger A_k |\psi\rangle$ is independent of k , and, therefore, the two conditions (1) and (2) are the same. Using Eqs. (20) and (21), we find

$$\langle\psi|(\prod_{k=1}^n |0\rangle_k\langle 1|)|\psi\rangle = \cos\theta \sin\theta, \quad (22)$$

$$\left(\prod_{k=1}^n \langle\psi|(|1\rangle_k\langle 1|)^{n/2}|\psi\rangle\right)^{1/n} = \sin^2\theta. \quad (23)$$

If $|\cos\theta| > |\sin\theta|$, it can be seen that both entanglement conditions are satisfied, indicating the presence of entanglement. Alternatively, one can choose $A_k = |1\rangle_k\langle 0|$, which implies that $A_k^\dagger A_k = |0\rangle_k\langle 0|$. Then the left-hand sides of Eqs. (1) and (2) are unchanged and given again by Eq. (22), while the right-hand sides become $\cos^2\theta$. In this case, entanglement is detected if $|\sin\theta| > |\cos\theta|$. This choice of A_k thus complements the one given in Eq. (21). These two choices detect entanglement in the state in Eq. (20) for all values of θ , except for the case of $\cos\theta = \sin\theta$. For $\sin 2\theta \leq 1/\sqrt{2^{n-1}}$ and n odd, the state in Eq. (20) does not violate any n -party Bell inequalities for correlation functions containing two dichotomic observables per local measurement station [23], a set of inequalities that includes the Mermin-Klyshko inequalities [24, 25, 26]. Therefore, for this state the conditions (1) and (2) are stronger criteria for entanglement detection than those coming from these Bell inequalities. The entanglement in the state (20) also eludes detection by all four spin squeezing inequalities derived in Refs. [12, 13], which include those presented in Refs. [8, 9, 10] as particular cases.

If the state has one spin flipped with respect to the rest

$$|\psi\rangle = \cos\theta|1\rangle \otimes |0\rangle^{\otimes(n-1)} + \sin\theta|0\rangle \otimes |1\rangle^{\otimes(n-1)}, \quad (24)$$

then by choosing

$$A_1 = |1\rangle_1\langle 0|, \quad A_k = |0\rangle_k\langle 1|, \quad k > 1, \quad (25)$$

one can readily find that entanglement is detected for $|\cos\theta| > |\sin\theta|$. Generalization to cases where more spins are flipped is straightforward.

Conditions (1) and (2) are also robust against noise. It is not difficult to verify that for the state

$$\rho = p|\psi\rangle\langle\psi| + (1-p)|0\rangle^{\otimes n}\langle 0|, \quad 0 < p < 1, \quad (26)$$

where $|\psi\rangle$ is given by Eq. (20), they work the same as discussed above. Interestingly, this holds no matter how large the amount of noise is, that is how close p is to zero, because p appears in the same way on both sides of the inequality and cancels out. One can assume a more general type of noise

$$\rho = p|\psi\rangle\langle\psi| + (1-p)\frac{I}{2^n}, \quad 0 < p < 1, \quad (27)$$

where I is the unity operator. With the choice of A_k as in Eq. (21), condition (1) yields

$$|\cos\theta\sin\theta| > \sin^2\theta + \frac{1-p}{2p}. \quad (28)$$

This inequality becomes increasingly difficult to satisfy as p decreases, and impossible to satisfy for $p \leq 1/3$. Therefore, for states that are not too noisy, our conditions can still detect entanglement.

3.1.2 A partially separable state

The state (20) is a genuinely multipartite entangled state. We give now some examples to see how the two conditions (1) and (2) work with a partially separable state. Consider again an ensemble of n spin $\frac{1}{2}$ particles, split into two groups of l and $(n-l)$ spins, each being in a generalized GHZ state

$$|\psi\rangle = [\cos\theta_1|0\rangle^{\otimes l} + \sin\theta_1|1\rangle^{\otimes l}] \otimes [\cos\theta_2|0\rangle^{\otimes(n-l)} + \sin\theta_2|1\rangle^{\otimes(n-l)}]. \quad (29)$$

Choosing A_k as in Eq. (21), the inequalities in Eqs. (1) and (2) become

$$|\cos\theta_1\sin\theta_1\cos\theta_2\sin\theta_2| > [(\sin\theta_1)^{2l}(\sin\theta_2)^{2(n-l)}]^{1/n}, \quad (30)$$

$$\begin{aligned} |\cos\theta_1\sin\theta_1\cos\theta_2\sin\theta_2| &> \left(\frac{n-l}{n}\right)^{n/2} \cos^2\theta_1 \sin^2\theta_2 \\ &+ \left(\frac{l}{n}\right)^{n/2} \cos^2\theta_2 \sin^2\theta_1 + \sin^2\theta_1 \sin^2\theta_2, \end{aligned} \quad (31)$$

respectively. Note that the two conditions now behave differently. It is apparent from the above equations that entanglement cannot be detected if $\sin\theta_1 = 0$ or $\cos\theta_1 = 0$, which we exclude from further consideration. Since the inequalities (30) and (31) are rather involved, it is instructive to examine some special cases.

For $l = 1$ and $n = 3$, and $\sin\theta_1 = \cos\theta_1 = \frac{1}{\sqrt{2}}$, they become

$$|\cos\theta_2| > (4|\sin\theta_2|)^{1/3}, \quad (32)$$

$$|\cos\theta_2\sin\theta_2| > 1.09|\cos\theta_2\sin\theta_2| + (1.24|\sin\theta_2| - 0.44|\cos\theta_2|)^2. \quad (33)$$

Obviously, when θ_2 is close enough to 0 or π , the first inequality, Eq. (32), is satisfied meaning it can detect entanglement in the state, while there exists no θ_2 for which the second inequality, Eq. (33), is satisfied.

For $l = 2$ and $n = 4$, the two inequalities (30) and (31) become

$$|\cos \theta_1 \sin \theta_1 \cos \theta_2 \sin \theta_2| > |\sin \theta_1 \sin \theta_2|, \quad (34)$$

$$|\cos \theta_1 \sin \theta_1 \cos \theta_2 \sin \theta_2| > \frac{1}{2}(\cos \theta_1 \sin \theta_2 - \cos \theta_2 \sin \theta_1)^2 + 2 \sin^2 \theta_1 \sin^2 \theta_2, \quad (35)$$

respectively. It can be seen that the inequality (34) cannot be fulfilled for any values of θ_1 and θ_2 . Regarding the second inequality, we set for simplicity $\cos \theta_1 \sin \theta_2 = \cos \theta_2 \sin \theta_1$, to make it become $\cos^2 \theta_2 > 2 \sin^2 \theta_2$, which clearly can be fulfilled with θ_2 in the neighborhood of 0 and π . Thus the situation is opposite to that occurring in the case of $l = 1$ and $n = 3$ discussed above in that the second condition, not the first one, does better at detecting entanglement.

In the limit of large n but fixed l , Eqs. (30) and (31) can be brought approximately to comparable forms

$$|\cos \theta_2| > |\sin \theta_2| \frac{1}{|\cos \theta_1 \sin \theta_1|}, \quad (36)$$

$$|\cos \theta_2| > |\sin \theta_2| \frac{1}{|\cos \theta_1 \sin \theta_1|} \left[e^{-l/2} + (1 - e^{-l/2}) \sin^2 \theta_1 \right], \quad (37)$$

where in going from Eq. (30) to Eq. (36), we made the replacement $(\frac{\sin \theta_2}{\sin \theta_1})^{\frac{n-2l}{n}} \rightarrow \frac{\sin \theta_2}{\sin \theta_1}$, while and in going from Eq. (31) to Eq. (37), we used the relation $(\frac{n-l}{n})^{n/2} = e^{-l/2} + O(\frac{1}{n})$ and dropped the vanishing second term in the right-hand side of Eq. (31). Two comments can be made regarding the inequalities (36) and (37). First, there exist θ_1 and θ_2 for which both or one of them are satisfied, meaning entanglement is detected. Second, since $\sin^2 \theta_1 < 1$, the extra factor in Eq. (37) is less than unity with the result that the condition in Eq. (37) is more sensitive to entanglement than that in Eq. (36) in the sense that there exist ranges of the parameters θ_1 and θ_2 for which condition (37) can detect entanglement while condition (36) cannot.

An estimate of how little entanglement in a multipartite system is detectable by condition (1) can be gained by studying the state

$$|\psi\rangle = \prod_{i=1}^{l \otimes} [\cos \theta_i |0\rangle_i + \sin \theta_i |1\rangle_i] \otimes [\cos \theta |0\rangle^{\otimes(n-l)} + \sin \theta |1\rangle^{\otimes(n-l)}], \quad (38)$$

where l parties are separable, while the remaining parties are in a GHZ state. For this state, condition (1) gives us

$$|\cos \theta| > \frac{1}{\prod_{i=1}^l |\cos \theta_i (\sin \theta_i)^{1-2/n}|} |\sin \theta|^{(n-2l)/n}. \quad (39)$$

Since the denominator is less than one, the inequality can be satisfied when $l < n/2$. Though the number of separable parties has to be smaller than half the total number of parties for entanglement to be detected, using condition (1) to look for entanglement is clearly less labor-intensive than using a bipartite condition to check every possible pairwise separation, especially in the case of large n .

3.1.3 A mixed state

Consider the n -party state

$$\rho = \frac{1}{n} \sum_{i=1}^n |\psi_i\rangle \langle \psi_i|, \quad (40)$$

where

$$|\psi_i\rangle = [\cos\theta_i|0\rangle + \sin\theta_i|1\rangle] \otimes [\cos\theta|0\rangle_i^{\otimes(n-1)} + \sin\theta|1\rangle_i^{\otimes(n-1)}], \quad (41)$$

$|0\rangle_i^{\otimes(n-1)}$ and $|1\rangle_i^{\otimes(n-1)}$ being states where the spin i is excluded. $|\psi_i\rangle$ represents a state where the spin i is separated, while the remaining spins are in a generalized GHZ state. Though ρ is a statistical mixture of bipartite separable states, there is no overall bipartite splitting with respect to which the state is separable. Choosing A_k as in Eq. (21), the inequalities (1) and (2) are

$$|\cos\theta \sin\theta \sum_{i=1}^n \cos\theta_i \sin\theta_i| > \left[\prod_{i=1}^n [\sin^2\theta_i + (n-1)\sin^2\theta] \right]^{1/n}, \quad (42)$$

$$\begin{aligned} |\cos\theta \sin\theta \sum_{i=1}^n \cos\theta_i \sin\theta_i| &> \left(\frac{n-1}{n} \right)^{n/2} \sin^2\theta \sum_{i=1}^n \cos^2\theta_i \\ &+ \left(\frac{1}{n} \right)^{n/2} \cos^2\theta \sum_{i=1}^n \sin^2\theta_i + \sin^2\theta \sum_{i=1}^n \sin^2\theta_i. \end{aligned} \quad (43)$$

It is instructive to consider the case of very large n , $\sin\theta_1 = \cos\theta_1 = \frac{1}{\sqrt{2}}$, $\sin\theta_i = 0$ for $i \geq 2$, for which these inequalities simplify greatly to become

$$|\cos\theta| > 2(n-1)|\sin\theta|, \quad (44)$$

$$|\cos\theta| > \left[\frac{2}{\sqrt{e}} \left(n - \frac{1}{2} \right) + 1 \right] |\sin\theta|, \quad (45)$$

respectively. It can be seen that both inequalities can be satisfied if $|\cos\theta|$ is sufficiently close to one and both would perform worse as the number of parties n increases, the first more so than the second.

3.2 Continuous-variable systems

Consider an n -mode squeezed vacuum field state

$$|\psi\rangle = \sqrt{1-x^2} \sum_{m=0}^{\infty} x^m |m\rangle^{\otimes n}, \quad (46)$$

where $0 < x < 1$. For the choice of A_k

$$A_k = a_k, \quad A_k^\dagger A_k = a_k^\dagger a_k, \quad (47)$$

a_k being the annihilation operator of the field mode k , the two conditions (1) and (2) are identical. One finds that

$$\langle\psi| \left(\prod_{k=1}^n a_k \right) |\psi\rangle = \frac{1}{x} (1-x^2) \sum_{m=0}^{\infty} x^{2m} m^{n/2}, \quad (48)$$

$$\left(\prod_{k=1}^n \langle\psi| (a_k^\dagger a_k)^{n/2} |\psi\rangle \right)^{1/n} = (1-x^2) \sum_{m=0}^{\infty} x^{2m} m^{n/2}. \quad (49)$$

A comparison of Eqs. (48) and (49) shows that the inequalities (1) and (2) are satisfied for any value of x in the range $0 < x < 1$. That is to say, these conditions can always detect entanglement in the multimode squeezed vacuum state.

We consider now an example of continuous variable systems where the two conditions (1) and (2) work differently, namely a modified four-mode squeezed vacuum state

$$|\psi\rangle = \sqrt{1-x^2} \sum_{m=0}^{\infty} x^m |m\rangle_1 |m\rangle_2 |m+1\rangle_3 |m+1\rangle_4, \quad (50)$$

where $0 < x < 1$. Choosing A_k as in Eq. (47), we find that

$$\langle\psi|(\prod_{k=1}^4 a_k)|\psi\rangle = \frac{2x}{(1-x^2)^2}, \quad (51)$$

$$\left(\prod_{k=1}^4 \langle\psi|(a_k^\dagger a_k)^2|\psi\rangle\right)^{1/4} = \frac{x(1+x^2)}{(1-x^2)^2}, \quad (52)$$

$$\frac{1}{16} \langle\psi|\left(\sum_{k=1}^4 a_k^\dagger a_k\right)^2|\psi\rangle = \frac{1}{(1-x^2)^2} \frac{1}{4} (x^4 + 6x^2 + 1), \quad (53)$$

where we have made use of the relations $\sum_{m=0}^{\infty} x^{2m} = \frac{1}{1-x^2}$, $\sum_{m=0}^{\infty} x^{2m} m = \frac{x^2}{(1-x^2)^2}$, and $\sum_{m=0}^{\infty} x^{2m} m^2 = \frac{x^2(1+x^2)}{(1-x^2)^3}$. From Eqs. (51) and (52) it can be inferred that inequality (1) is satisfied for all x , meaning it always detects entanglement. A comparison of Eq. (51) and (53) however shows that inequality (2) can be used for entanglement detection only for $x \gtrsim 0.1397$.

4 Conclusion

We have presented two sufficient conditions for determining when multipartite states are entangled. These conditions are quite flexible, because the operators appearing in them can be chosen to best match the systems being considered. The conditions can be used to test for entanglement in discrete systems, continuous-variable systems, or mixtures of the two.

We have already seen that similar conditions for testing bipartite entanglement have proven useful in detecting entanglement in a variety of systems, including interacting spin systems and a collection of atoms interacting with the electromagnetic field. We expect that the conditions derived here for multipartite systems will prove similarly useful.

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